$$k^{**}=\frac{l_1l_2(F+A^2H)+ABH(l_2-l_1)+B^2H}{Fl_2^2+H(Al_2-B)^2}$$

Figure 2 depicts the surface $D_{v_1}=f(k, v)$. When the velocities of motion approach the critical value v^* defined by the expression (1.2), the dispersion increases without bounds as the value of the leading Hurwitz determinant of the system appearing in the denominator of the expression for dispersion tends to zero.

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Trnaslated by L. K.

UDC 539.95

ON THE CONSTRUCTION OF PLANE STATIONARY SOLUTIONS OF EQUATIONS FOR NONEQUILIBRIUM MAGNETIZED PLASMA

PMM Vol. 40, № 5, 1976, pp. 813-822

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(Received January 14, 1976)

A method for determining stationary two-dimensional distribution of the electric current and electron temperature in nonequilibrium magnetized plasma is developed with heat conduction and convestion taken into account. Solution is derived in the form of asymptotic expansions in a small parameter. Derivation of the zero approximation for the external and internal expansions is investigated. The problem of current distribution in a channel with infinite electrodes is considered as an example.

1. If heat conduction and convestion are neglected, the problem of stationary distribution of current in nonequilibrium plasma can be reduced to the problem of continuous media electrodynamics with a nonlinear dependence of electrical conductivity and of the Hall parameter (Ω) on the modulus of the vector of electric current density [1]. In the plane case this problem reduces to a quasi-linear equation of second order for the function of current or electrical potential [2-4] (Eq. (3.1) below). When the Hall parameter exceeds a certain value which coincides with the Hall parameter

critical value for ionization instability, this equation changes from the elliptical to the hyperbolic kind [3-5]. Transformation of the hodograph for Eq. (3.1) was used in [2] for $\Omega = 0$ and in [6] for $\Omega \neq 0$. Solutions of two problems in the elliptic region were derived in [6], and a numerical solution in the ellipticity regions appears in [3].

In the case of small degree of ionization and constant Hall parameter the problem of current distribution in the hyperbolic region can be reduced to a first order equation whose particular solution was used in [4] for the channel of an MHD generator with sectioned electrodes. A similar solution can be derived by the method of characteristics [5]. It was shown in [7] that solutions obtained in [4, 5] are nonevolutionary. Since heat conduction was not taken into consideration in [3-6], it was not possible to satisfy the conditions for the electron temperature. Experimental investigations of current distribution carried out in the channel of an MHD generator and in a discharge with external electric field are described in [8, 9] and [4, 10] respectively. When the Hall parameter exceeds the critical value, the current distribution has the form of prolate layers (streamers) along the middle current. The time of streamer development is considerably shorter than that of the ionization instability.

The solution of the problem of stationary plane current distribution in nonequilibrium plasma is considered here with allowance for heat conduction and convection.

In the analysis of the plane stationary problem we take the coordinates x and y as the independent variables, and introduce the dimensionless parameters

$$x^+ = \frac{x}{b}, \quad y^+ = \frac{y}{b}, \quad \sigma^+ = \frac{\sigma(n_e, T_e)}{\sigma^*(n_e^*, T_e^*)}, \quad \lambda^+ = \frac{\lambda_e(n_e, T_e)}{\lambda_e^*(n_e^*, T_e^*)} \tag{1.1}$$

$$\tau^+ = \frac{\tau_e(n_e, T_e)}{\tau_e^*(n_e^*, T_e^*)}, \quad \mathbf{j}^+ = \frac{\mathbf{j}}{j^*}, \quad \Psi^+ = \frac{\Psi}{j^* b}, \quad n^+ = \frac{n_e}{n_e^*}$$

$$F_-^+ = \frac{F_- \sigma^*}{j^{*2}}, \quad \varepsilon = \frac{kT_e^*}{I}, \quad \alpha = \frac{n_e^*}{n_g}, \quad a_\tau = \frac{d \ln \tau^+}{d \ln n^+}$$

$$\Lambda = \frac{\sqrt{T_e^* \lambda_e^* \sigma^*}}{j^* b}$$

where $\mathbf{j}$ is the density of the electric current; T_e is the temperature of electrons and n_e their concentration; σ and λ_e are the coefficients of electrical and thermal conductivities of electrons, respectively; b is a characteristic dimension (e.g., distance between electrodes); I is the ionization potential of the additive and n_g is the initial concentration of the additive atoms. The transfer of energy from electrons to heavy particles is taken into account by function F . The stream function $\Psi(x, y)$ is related to the electric current density by formula

$$j_x = \frac{\partial \Psi}{\partial y}, \quad j_y = -\frac{\partial \Psi}{\partial x} \tag{1.2}$$

When the characteristic density of electric current j^* is specified, the characteristic temperature T_e^* is determined by the equation

$$F_- = {}^3/_2 \delta k T_e^* n_e^* (T^*) [\tau_e^*]^{-1} = j^{*2} [\sigma^*]^{-1} \tag{1.3}$$

where $\delta = 2m_e / m_a$, τ_e is the time of energy transfer from electrons to heavy particles, and k is the Boltzmann constant. The electron concentration and their temperature are assumed to be related by the equation $n_e = n_e(T_e)$ (e.g., by the Saha equation) and $T_e \gg T_a$.

If the electron heat conduction is determined only by the conductivity component and the effect of the magnetic field on it is negligible, and T_e and n_e are linked by the Saha equation, the equation of electrodynamics and of energy for electrons (the complete system of equations appears in [11] are of the form

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \left[\Omega\,\frac{\partial \ln n}{\partial y} - (1 + a_\tau)\,\frac{\partial \ln n}{\partial x}\right] \times \frac{\partial\Psi}{\partial x} - \left[\Omega\,\frac{\partial \ln n}{\partial x} + (1 + a_\tau)\,\frac{\partial \ln n}{\partial y}\right]\frac{\partial\Psi}{\partial y} = 0. \tag{1.4}$$

$$\frac{\alpha n^2}{1 - \alpha n} = aT^{3/2}\exp(-\varepsilon^{-1}T^{-1})$$

$$\Lambda^2\left(\frac{\partial}{\partial x}\lambda\,\frac{\partial T}{\partial x} + \frac{\partial\lambda}{\partial y}\frac{\partial T}{\partial y}\right) - \Lambda\left(\frac{\partial \ln n}{\partial \ln T} - \frac{3}{2}\right)\left(\frac{\partial\Psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial T}{\partial y}\right) = \frac{nT}{\tau} - \sigma^{-1}\left[\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2\right]$$

where a is a known constant, $\Omega = e / m_e B\tau_e = \Omega^*\tau$ is the Hall parameter for the electrons, B is the magnetic field induction, and $\mathbf{B} = (0,\ 0,\ B)$. Since in what follows only dimensionless variables are used, the superscript plus is here and subsequently omitted. Unlike in [11] a more complete representation of the convection term in the third equation of system (1.4), which is valid at complete ionization of the additive $(\alpha \sim 1)$ is used here.

We seek the solution of system (1.4) in some region G, whose part G_1 of its boundary corresponds to perfectly conducting electrodes and part G_2 represents insulators. At the electrodes and inlulators the following conditions must be, respectively, satisfied:

$$\mathbf{E}\cdot\boldsymbol{\chi}^\circ = 0, \quad \mathbf{j}\cdot\mathbf{n}^\circ = 0 \tag{1.5}$$

where χ° is a vector in the electrode plane and $\mathbf{n}^\circ$ is a vector normal to the insulator wall surface.

In addition, along the whole boundary $G = G_1 \cup G_2$ the relationship

$$L(T,\ \nabla T,\ j) = 0 \tag{1.6}$$

which links the electron temperature, its gradient, and the normal component of the electric current density, must be specified.

The functional form of formula (1.6) at the electrodes and insulators may be different. However, all further investigations can be carried out without specifically defining the form of L.

For the majority of real problems $\Lambda \ll 1$. Since the parameter Λ appears at the higher derivative, the solution of the input problem can be derived by the method of singular perturbations [12,13] according to which the general solution may be represented in the form of the external expansion

$$T = \sum_{k=0}^{\infty} \Lambda^k T_k(x, y), \; n = \sum_{k=0}^{\infty} \Lambda^k n_k(x, y) \qquad (1.7)$$

$$\Psi = \sum_{k=0}^{\infty} \Lambda^k \Psi_k(x, y)$$

and of the internal one

$$T = \sum_{k=0}^{\infty} \Lambda^k \theta_k(x^*, y^*), \quad \mathbf{j} = \sum_{k=0}^{\infty} \Lambda^k \mathbf{j}_k(x^*, y^*) \qquad (1.8)$$

$$\mathbf{E} = \sum_{k=0}^{\infty} \Lambda^k \mathbf{E}_k(x^*, y^*)$$

where $x^* = x / \Lambda$, $y^* = y$ or $x^* = x$, and $y^* = y / \Lambda$.

Expansion of the from (1.7) makes it possible to satisfy the boundary conditions for the electrodynamic variables (1.5); to satisfy conditions (1.6) it is necessary to use the internal expansion (1.8).

2. Internal expansions used for deriving plane solutions in a closed region may be of different forms, viz. solutions of the kind of boundary layer at the walls; solutions of the kind of stationary ionization and a recombination of the additive in the inner channel regions, and solutions of the kind of two-temperature boundary layer for which an allowance for the finiteness of heavy particle temperature is essential. Solutions of the last kind occur when solutions at the current boundary region $(T \gg T_a)$ are joined with the region of equilibrium plasma $(T \sim T_a)$.

We assume for the sake of definiteness that the electrode plane lies in the xz-plane. Introducing the independent variables $x^* = x$ and $y^* = y / \Lambda$), we substitute (1.8) into system (1.4), equate terms of like order with respect to Λ , and obtain (for the zero terms in expansion (1.8)) the following system of equations:

$$\frac{\partial E_{0x}}{\partial y^*} = 0, \quad \frac{\partial j_{0y}}{\partial y^*} = 0 \qquad (2.1)$$

$$\frac{\partial}{\partial y^*} \lambda(\theta_0) \frac{\partial \theta_0}{\partial y^*} + U_y(\theta_0) \frac{\partial \theta_0}{\partial y^*} + F(\theta_0) = 0$$

$$F = j_{0y}^2 \sigma^{-1}(\theta_0)[1 + \Omega^2(\theta_0)] - F_-(\theta_0), \quad U_y = -j_{0y}\left(\frac{\partial \ln n_0}{\partial \ln T_0} - \frac{3}{2}\right)$$

The first and second equations of system (2.1) imply that within the boundary layer E_{0x} is identically zero (since at the electrode $E_x = 0$), and j_{0x} is an arbitrary function that depends on x^* and is to be determined by the construction of the complete problem (e.g., the potential difference between electrodes or the equation of the electric circuit containing the considered channel can be specified). In such case the derivation of solution for the boundary layer reduces to the integration of the equation of energy for the electrons, which is solved for $y^* \to \infty$ with the following boundary conditions:

$$\theta_0(x^*, y^* \to \infty) \to T_0(x, y = 0), \quad \frac{\partial \theta_0}{\partial y^*} \to 0 \qquad (2.2)$$

The second boundary condition is the condition at the electrode $L\left(\theta_0\,(x^*,\, y^* = 0),\ \frac{\partial\theta_0}{\partial y^*},\, j_{0y}\right) = 0$. We assume that the specified distribution of electron temperature over the electrode $T_1(x)$ is

$$L \equiv \theta_0\,(x^*,\, 0) - T_1\,(x) = 0 \tag{2.3}$$

It follows from (2.2) that the external solution at $y = 0$ must correspond to a singular point of the third equation of system (2.1), i.e., it is the zero of function F. Inside the boundary layer the temperature specified at the electrode asymptotically reaches the temperature at the core of the stream.

For the boundary layer at the insulator wall whose plane is also in the xz-plane it is necessary to set $U_y = 0$ and

$$F\,(\theta_0) = \sigma\,(\theta_0)\, E_{0x}{}^2 - F_-\,(\theta_0) \tag{2.4}$$

This equation can be integrated

$$y^* = \pm \int\limits_{T_2(x)}^{\theta_0} \frac{\lambda\,(S)\, dS}{\sqrt{2 \int\limits_{S}^{T_0\,(x,\, y=0)} \lambda F\,(S)\, dS}} + \text{const}$$

where $T_2\,(x)$ is the specified temperature distribution on the insulation wall.

Calculations can only be carried out for a specific plasma composition. Thus for argon with the addition of cesium function F has in the temperature range $10^3 - 10^4\,^\circ$K a single zero T_0 for any value of the variable $j_{0y}\,(E_{0x})$. The related equilibrium point $(T_0,\, 0)$ is a saddle point. The solution in the boundary layer corresponds to the separatrix oriented in the direction of decreasing (increasing) temperature when the wall temperature is higher (lower) than that at the boundary of the boundary layer (when $y^* \to \infty$). Since U_y at the anode and the cathode differs at least in their sign, the boundary layers at these are different.

The problem of stationary ionization discontinuity surfaces (ID) also reduces to the integration of energy equations. In that case it is always possible to turn the input coordinate system $(x,\, y) \to (x^*,\, y^*)$ so as to have the y^*-axis merge with the normal to the wave surface. In the new system of coordinates all dependent quantities are functions of y^*. The first two equations of system (2.1) state the conservation of the tangent component of the electric field E_{0x} and of the normal component of the electric current j_{0y}. Function F is of the form

$$F = j_{0y}{}^2\sigma^{-1}\,(1 + \Omega^2) - 2\Omega j_{0y} E_{0x} + \sigma E_{0x}{}^2 - F_- \tag{2.5}$$

In considering stationary ID we assume, unlike in [14], that the total current j_1 and the angle φ_1 between the normal to the wave surface (the y^*-axis) the vector j_1 are specified upstream of the wave.

The kind of equilibrium points, their number and subscript depend on j_1, φ_1 and the Hall parameter $\Omega_1 = \Omega_1\,(j_1,\, B)$ upstream of the wave. These are linked to the current density j_{0y} and the electric field E_{0x} by the relationship

$$j_{0y} = j_1 \cos\varphi_1, \quad E_{0x} = j_1\sigma^{-1}\,(\sin\varphi_1 + \Omega_1 \cos\varphi_1) \tag{2.6}$$

Function F has always a single zero for any j_1, φ_1 and Ω_1, which is determined by the equation $j_1^2\sigma^{-1} = F_-(T_{01})$ When $\Omega_1 < \Omega_-$ ($\Omega_-(j_1)$ generally does not coincide with the critical Hall parameter for ionization instability) there are no other zeros except T_{01} for any angle φ_1. When $\Omega > \Omega_1$ function F has two more zeros T_{02} and T_{03} in the particular angle interval $\varphi_- < \varphi < \varphi_+$ ($\varphi_\mp = \varphi_\mp(j_1, \Omega_1)$)and there always exists a region of angles φ_1 in the interval (φ_-, φ_+) where points (T_{01}, 0) and (T_{03}, 0) are saddle points, and (T_{02}, 0) is either a nodal or focal point, or a center, depending on the values of $U_y(T_{02})$ ($T_{01} < T_{02} < T_{03}$).

Besides the solutions of system (2.1) considered in [14,15] system (2.1), (2.5) admits yet one more solution of the kind of standing layer wave (ionization and recombination wave, with the state upstream of the ionization wave is the same as the state downstream of the recombination wave). A closed separatrix which passes through points (T_{01}, 0) and (T_{03}, 0) corresponds to it in the phase plane. The conditions of existence of the layer wave can be formulated as follows:

$$\oint \lambda(T) U_y(T) \frac{dT}{dy^*} dT = 0$$

$$\int_{T_{01}}^{T_{03}} \lambda(T)\left[U_y(T) \frac{dT}{dy^*} + F(T)\right] dT = 0$$

For a specified magnetic field induction a layer wave can only be initiated for particular values of angle φ_1 and current density j_1. This property will be used in the construction of the general solution of the problem.

The two-temperature boundary layer arises at the interface of region of high ($T \gg T_a$) and low ($T \sim T_a$, $n \to 0$) electrical conductivity. To construct such boundary layer it is necessary to use besides the equation of energy for the electrons, the equation of energy for atoms, which in the region of nonequilibrium plasma ($T \gg T_a$) (in the absence of convective transport) can be represented in the form

$$\Lambda_1^2 \nabla\lambda_a(T_a)\nabla T_a + F_-(T_0(x, y)) = 0 \tag{2.7}$$

where $\Lambda_1 = \sqrt{T_e^* \lambda_e^* \sigma^*}\,(j^* b)^{-1}$, and λ_a is the dimensionless coefficient of atomic thermal conductivity in the equilibrium region ($T \sim T_a$, $n \to 0$, $\sigma \to 0$)

$$\nabla[\lambda(T_a) + \lambda_a(T_a)]\nabla T_a = 0 \tag{2.8}$$

In constructing the two-temperature boundary layer it is possible to use the approximate procedure by expressing function F_- in the second of Eqs. (2.1) in the form

$$F_- = n\tau^{-1}(T)(T - T_{a3})$$

where T_{a3} is the unknown temperature of atoms at the equilibrium zone boundary. This unknown temperature is determined by the solution of Eqs. (2.7) and (2.8) in its zone, and by the conditions of merging of heat fluxes at the interface (the atom temperature may not have a boundary layer, while the latter exists for the electron temperature)

$$\Lambda_1^2 \lambda_a(T_{a3}) \left(\frac{\partial T_a}{\partial n^\circ}\right)_{-0} - \sqrt{2 \int_{T_{a3}}^{T_0} \lambda(S) F(S)\, dS} =$$

$$\Lambda_2^2 [\lambda (T_{a3}) + \lambda_a (T_{a3})] \left(\frac{\partial T_a}{\partial n^\circ}\right)_{+0}$$

where $\partial / \partial \mathbf{n}$ are derivatives along the normal to the discontinuity surface, $\mathbf{n}^\circ$ is the unit vector along the normal in the direction of equilibrium plasma, and

$$\Lambda_2 = \sqrt{T_e^* \sigma^* (\lambda_e^* + \lambda_a^*)}\, (j^* b)^{-1}.$$

Conditions of heat exchange must be used in the regions where equilibrium plasma zones adjoin the channel walls.

3. Equations of the external expansion can be obtained by substituting (1.7) into system (1.4) and equating terms of like order with respect to Λ_*. The system of zero approximations formally corresponds to the disregard of convective and conductive heat transfer in the equation of energy for electrons, and all observations appearing in the introduction are valid in this case. The system can be reduced to a single equation for function $\Psi_0(x, y)$, as was done in [4, 5]

$$a_{11}\frac{\partial^2 \Psi_0}{\partial x^2} + a_{12}\frac{\partial^2 \Psi_0}{\partial x\, \partial y} + a_{22}\frac{\partial^2 \Psi_0}{\partial y^2} = 0 \tag{3.1}$$

$$a_{11} = 1 - \frac{1 + a_\tau}{j^2} b_1 \left(\frac{\partial \Psi_0}{\partial x}\right)^2 - \frac{\Omega b_1}{j^2}\frac{\partial \Psi_0}{\partial x}\frac{\partial \Psi_0}{\partial y}$$

$$a_{12} = \frac{b_1}{j^2}\left\{\Omega\left[\left(\frac{\partial \Psi_0}{\partial x}\right)^2 - \left(\frac{\partial \Psi_0}{\partial y}\right)^2\right] - 2(1 + a_\tau)\frac{\partial \Psi_0}{\partial y}\frac{\partial \Psi_0}{\partial x}\right\}$$

$$a_{22} = 1 - \frac{b_1}{j^2}\left[(1 + a_\tau)\left(\frac{\partial \Psi_0}{\partial y}\right)^2 - \Omega\,\frac{\partial \Psi_0}{\partial x}\frac{\partial \Psi_0}{\partial y}\right]$$

$$b_1 = \left[\frac{3}{2} + (\varepsilon T_0)^{-1}\right]\left[\frac{5}{2} + (\varepsilon_0 T_0)^{-1} + \frac{\alpha n_0}{2}(1 - \alpha n_0)^{-1}\right]^{-1}$$

$$j^2 = \left(\frac{\partial \Psi_0}{\partial x}\right)^2 + \left(\frac{\partial \Psi_0}{\partial y}\right)^2$$

where Ω, b_1, a_τ and T_0 are known functions of j^2. Equation (3.1) is equivalent to a system of two first order equations with respect to components of the current density vector

$$\frac{\partial R}{\partial x} + Z(R)\frac{\partial R}{\partial y} = 0$$

$$R = \begin{Vmatrix} j_y \\ j_x \end{Vmatrix}, \quad Z = \begin{Vmatrix} \frac{a_{12}}{a_{11}} & -\frac{a_{22}}{a_{11}} \\ 1 & 0 \end{Vmatrix} \tag{3.2}$$

The equation of characteristics of system (3.2) is

$$\frac{dy^\pm}{dx} \equiv v_\pm = \frac{a_{12} \pm b_1\sqrt{\Omega^2 - \Omega_*^2}}{2a_{11}} \tag{3.3}$$

$$\Omega_* = \frac{2}{b_1}\sqrt{1 - b_1(1 + a_\tau)} = \sqrt{\left(\frac{\partial \ln k_-}{\partial \ln n_0}\right)^2 - \left(\frac{\partial \ln \sigma}{\partial \ln n_0}\right)^2}$$

Equation (3.1) (or system (3.2)) can be either of the elliptic $(\Omega < \Omega_*)$ or the hyperbolic $(\Omega > \Omega_*)$ kind.

The quantity Ω_* coincides with the critical Hall parameter for nonequilibrium ionization [16].

When $\Omega > \Omega_*$ system (3.2) admits three classes of continuous solutions: a homogeneous solution ($R \equiv$ const), two singular solutions (simple waves or Riemann waves [17]), and a general solution. In the case of the singular solution the components of vector R are interconnected, i.e., $j_y = j_y(j_x)$. It corresponds to the degeneration of the image of the physical plane (x, y) on the hodograph plane (j_x, j_y): $\partial(j_x, j_y) / \partial(x, y) \equiv 0$. Singular solutions of system (3.2) may be represented in the form

$$y - v_{\pm}x = \Phi_{\pm}(j_x), \quad dj_y/dj_x = v_{\pm} \tag{3.4}$$

where Φ is an arbitrary function of j_x.

Attempts at obtaining an analytic representation of the general solution for an arbitrary dependence of coefficients a_{ij} on the electric current density were unsuccessful. However, for considerable Hall parameters the solution (zero approximation in the series expansion in the small parameter $1/\Omega$) can be constructed by the hodograph method [18].

In the hodograph plane (j_x, j_y) Eq. (3.1) may be written in the form

$$2\xi \frac{\partial^2\Phi}{\partial\xi\,\partial\eta} = \frac{\partial\Phi}{\partial\eta}, \quad \xi = j_x^2 + j_y^2, \quad \eta = \frac{j_y}{j_x} \tag{3.5}$$

Mapping onto the physical plane is effected by the transformations

$$x = \partial\Phi/\partial j_y, \quad y = -\partial\Phi/\partial j_x \tag{3.6}$$

The solution of Eq. (3.5) is obtained by the method of separation of variables, which yields

$$\begin{aligned} x &= \frac{\eta}{\sqrt{1+\eta^2}}\,(F_1(\eta) + F_2(\xi)) + \xi\eta\,\frac{dF_1}{d\eta} \\ y &= \frac{-1}{\sqrt{1+\eta^2}}\,(F_1(\eta) + F_2(\xi)) + \sqrt{1+\eta^2}\,\frac{dF_1}{d\eta} \end{aligned} \tag{3.7}$$

where F_1 and F_2 are arbitrary functions of η and ξ, respectively. In the particular case of $(F_1 \equiv 0)$ the general solution is of the form

$$\eta = -x/y, \quad \xi = F_2(x^2 + y^2) \tag{3.8}$$

and the streamlines are represented by concentric circles.

As an example, let us consider the model of weakly ionized plasma with a predominance of electron-atom collisions [4, 5], which makes it possible to obtain analytic formulas for simple waves. For such model $b_1 \to 1$, $a_\tau \to 0$, $\Omega_* \to 0$ and $\Omega =$ const. Setting $\Omega_* = 0$ we obtain

$$\frac{dy^+}{dx} = \frac{j_y}{j_x} \equiv \eta, \quad \frac{dy^-}{dx} = \frac{j_y - \Omega j_x}{j_x + \Omega j_y} = \frac{\eta - \Omega}{1 + \eta\Omega} \tag{3.9}$$

Equation (3.2) can be represented in the form of a system in terms of invariants

$$\left(\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\left(j^2 \exp\frac{2}{\Omega}\operatorname{arctg}\eta\right) = 0, \quad \left(\frac{\partial}{\partial x} + \frac{\eta - \Omega}{1 + \eta\Omega}\frac{\partial}{\partial y}\right)\eta = 0 \tag{3.10}$$

It follows from (3.9) that the plus characteristics coincide with electric current lines, while the minus characteristics coincide with the electric field lines and are straight lines. The equations of singular solutions (3.4) for the considered model can be integrated. Taking into account (3.9) we obtain the first singular solution

$$j = \Phi_+(y - \beta x), \quad j_y = \beta j_x, \quad j^2 = j_x^2 + j_y^2$$

where $\beta = \text{const}$. The second singular solution is

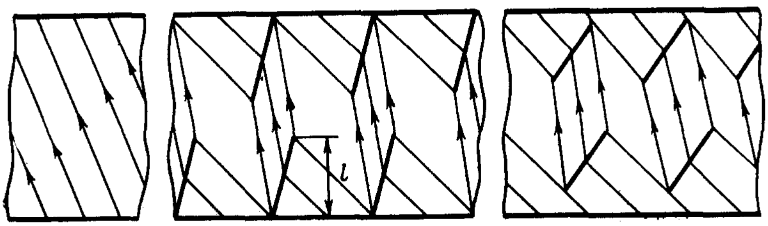
$$y - \frac{\eta - \Omega}{1 + \eta\Omega} x = \Phi_-(j_x), \quad j^2 = \text{const} \exp\left(-\frac{2}{\Omega} \operatorname{arctg} \eta\right)$$

The wave profile defined by solution (3.11) remains undistorted and current lines are parallel (solution of the kind of current filament with arbitrary current distribution in the filament). In the particular case of centralized wave ($\Phi_- \equiv 0$) of the second singular solution the current lines have the form of exponential spirals. Another method was used in [4, 5] for obtaining this solution

$$\sqrt{x^2 + y^2} = \text{const} \exp\left(\frac{1}{\Omega} \operatorname{arctg} \frac{x}{y}\right)$$

When $\Omega \to \infty$ the singular solution (3.13) coincides with the particular case of the general solution (3.8). The arbitrary current distribution in the filament in solution (3.11) is valid for the considered model (it is not generally valid, as can be tested by the direct substitution of (3.11) into (3.1)).

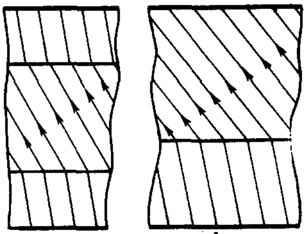
4. Let us consider a channel that is unbounded in the direction of the x and z-axes and bounded along the y-axis by two electrodes $(0 \leqslant y \leqslant 1, -\infty < x < +\infty)$ The electrodes are under a difference of potentials V.



a b c

Fig. 1

The external expansion region occupies almost the whole channel, except the narrow (of order Λ) regions of the boundary layer adjoining the electrodes. When $\Omega < \Omega_-$ the only continuous solution in the external expansion region is homogeneous; its current lines are shown in Fig. 1, a. If $\Omega > \Omega_-$ the homogeneous solution is unstable, and nonhomogeneous continuous solutions of system (3.2) may appear beside it. Such solutions represent homogeneous regions separated by a layer wave (Figs. 1, b and c). Depending on the number of parameters these waves can be separated in two classes: one-parametric and two-parametric Fig. 1, b and c. In the first case the structure is determined by a single linear dimension for instance by the



a b

Fig. 2

parameter l which is related to the difference of potentials V by formula

$$l = \frac{V - (j_2 \cos \gamma_2)^{-1}}{j_1 (\sigma_1 \cos \gamma_1)^{-1} - j_2 (\sigma_2 \cos \gamma_2)^{-1}}$$

where j_1, σ_1, γ_1, j_2, σ_2, and γ_2 are, respectively, the current density, the coefficient of electrical conductivity, and the angle between the current vector and the y -axis up- and downstream of the ionization wave. Since the two-parametric structures (Fig. 1, c) are defined by one more parameter besides l for instance by the width of the central current filament, hence the two-parametric nonhomogeneous solution is not unique. The obtained here structures (Figs 1, b and c) differ from those considered in [10]. Nonhomogeneous solutions exist in the specific interval $j_1 (\sigma_1 \cos \gamma_1)^{-1} \leqslant V \leqslant j_2(\sigma_2 \cos \gamma_2)^{-1}$ of variation of V, with the current flowing through a unit of the electrode surface constant and equal $j_1 \cos \gamma_1$.

For some plasma compositions and $\Omega > \Omega_-$ solutions containing standing waves, whose plane is parallel to that of the electrode (current lines are shown in Figs. 2, a and b), may occur in a particular range of electrode voltage. In such case nonhomogeneous solution of two kinds, viz. symmetric about the channel centre (Fig. 2, a) and nonsymmetric (Fig. 2, b) may appear besides the homogeneous solutions. Nonhomogeneous solutions with moving waves may also be present. Results presented in [11] make it possible to assume that waves corresponding to the auto-oscillation mode may develop on the background of furrent filaments, since the latter can be considered as channels with nonconducting walls.

Authors thank the participants of the seminar conducted by G. A. Liubimov for discussing this paper.

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Translated by J. J. D.

UDC 533. 951. 7.

SOLUTION OF THE FIRST ORDER QUASI-LINEAR EQUATION THAT DEFINES THE EVOLUTION OF PLASMA TURBULENCE

PMM Vol. 40. №5. 1976. pp. 823-833

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(Received January 16, 1976)

An asymptotic solution of the Cauchy problem is obtained for the first order quasi-linear equation. The field of characteristic curves is constructed. It is shown that for fairly considerable times the solution is discontinuous, but tends to a smooth stationary distribution. Numerical calculations obtained by the method of characteristics are presented. Results of the asymptotic and numerical analysis are in good agreement.